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Algebraic approach to non-central potentials

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Abstract
An algebraic approach to class of separable non-central Hamiltonians is presented. We show that the bound states of quantum systems under consideration are described by unitary representations of the $\mathfrak{so}(5)$ algebra.

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1. Introduction

Various types of relation between the Hamiltonian of a quantum system and operators of the enveloping algebra of a Lie algebra have been extensively investigated [1, 2] since the seminal work of Pauli [3]. Among the relations of interest is one that [4] associates the Hamiltonian H of a system with Casimir operators C of the Lie algebra $\mathfrak{g}$ restricted to some subspace $\mathcal{H}$ of carrier space, i.e.

$$H = f(C)|_{\mathcal{H}}. \tag{1.1}$$

In this case the algebra $\mathfrak{g}$ describes fixed energy states of a family of quantum systems with different potential strength. That is why the present algebra $\mathfrak{g}$ is called the 'potential algebra' [5].

Using the potential algebra, a number of quantum mechanical problems have been solved algebraically (see, e.g., [6–10] and references therein). Most of these potentials are either one dimensional or are central potentials. Hence, it is quite reasonable to ask whether one can also solve some non-central potential problems. The answer to the question is in the affirmative. We show that bound-state problems for the non-central potentials of the type

$$V^{(1)}(\mathbf{x}) = -\frac{\gamma}{r} - a_0^2\varepsilon_0\frac{n^2-\frac{1}{4}}{r^2\cos^2\theta} \tag{1.2}$$

and

$$V^{(2)}(\mathbf{x}) = -\frac{\gamma}{r} - a_0^2\varepsilon_0\frac{n^2-\frac{1}{4}}{r^2\sin^2\theta\sin^2\varphi}, \tag{1.3}$$

where r, θ, φ are spherical coordinates, and a_0 and ε_0 stand for the Bohr radius and the ground-state energy of the hydrogen atom, respectively, admit the Lie algebra $\mathfrak{so}(5)$ as the potential algebra. Namely,

$$H^{(i)} = -\frac{\gamma^2}{2(C+9/4)}\bigg|_{\mathcal{H}^{(i)}}, \qquad i = 1, 2, \tag{1.4}$$

where C is the second-order Casimir operator of $\mathfrak{so}(5)$, while $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ are subspaces occurring in the subalgebra reductions $\mathfrak{so}(5) \supset \mathfrak{so}(4) \supset \mathfrak{so}(2) \times \mathfrak{so}(2)$ and $\mathfrak{so}(5) \supset \mathfrak{so}(4) \supset \mathfrak{so}(3) \supset \mathfrak{so}(2)$, respectively.

2. Main idea

Let us start the discussion with the fact that (see, e.g., [11] and references therein) the symmetric (or, class 1) unitary irreducible representation (UIR) of $\mathfrak{so}(5)$ can be realized in the Hilbert space $\mathfrak{H}$ spanned by bound-state eigenfunctions corresponding to a fixed eigenvalue of the Coulomb Hamiltonian h in four dimensions, where

$$h = \frac{p^2}{2} - \frac{\gamma}{r}, \qquad \gamma > 0, \tag{2.1}$$

with $x = (x_1, x_2, x_3, x_4)$, $p = (p_1, p_2, p_3, p_4)$ and

$$r^2 = \sum_{i=1}^{4} x_i^2, \qquad p_i = -i\frac{\partial}{\partial x_i}, \quad i = 1, 2, 3, 4. \tag{2.2}$$

(We are using units with $M = \hbar = 1$.) As a prelude to this realization one introduces angular momentum and Runge–Lenz operators given by

$$L_{ij} = x_i p_j - x_j p_i \tag{2.3}$$

$$A_i = \frac{1}{2}(L_{ij} p_j + p_j L_{ij}) - \frac{\gamma x_i}{r}. \tag{2.4}$$

These operators satisfy the following commutation relations:

$$[L_{ij}, L_{kl}] = \mathrm{i}(\delta_{ik} L_{jl} + \delta_{jl} L_{ik} - \delta_{il} L_{jk} - \delta_{jk} L_{il}) \tag{2.5}$$

$$[L_{ij}, A_k] = \mathrm{i}(\delta_{ik} A_j - \delta_{jk} A_i) \tag{2.6}$$

$$[A_i, A_j] = -2\mathrm{i}h L_{ij} \tag{2.7}$$

$$[L_{ij}, h] = [A_i, h] = 0. \tag{2.8}$$

Defining now operators

$$L_{i5} = -L_{5i} \equiv \left(-\frac{1}{2h}\right)^{1/2} A_i \tag{2.9}$$

which are well defined in $\mathfrak{H}$ we obtain for $L_{\alpha\beta}, \alpha, \beta = 1, 2, \ldots, 5$, the commutation relations of the Lie algebra $\mathfrak{so}(5)$

$$[L_{\alpha\beta}, L_{\gamma\delta}] = \mathrm{i}(\delta_{\alpha\gamma} L_{\beta\delta} + \delta_{\beta\delta} L_{\alpha\gamma} - \delta_{\alpha\delta} L_{\beta\gamma} - \delta_{\beta\gamma} L_{\alpha\delta}). \tag{2.10}$$

Thus the symmetric UIR of $\mathfrak{so}(5)$ is realized in the Hilbert space $\mathfrak{H}$ of the bound-state wave functions $\Phi(x)$ corresponding to the fixed energy subspace, with inner product

$$(\Phi, \Phi') = \int_{R^4} \Phi^*(x)\Phi'(x)\, \mathrm{d}^4 x. \tag{2.11}$$

In this realization the representation operators are given by equations (2.3) and (2.4). If we compute the second-order Casimir operator

$$C = \frac{1}{2}\sum_{\alpha,\beta} L_{\alpha\beta}^2 \tag{2.12}$$

for this realization, it becomes

$$C = -\frac{9}{4} - \frac{\gamma^2}{2h}. \tag{2.13}$$

Next, imposing the reduction conditions, one can extract the corresponding non-central potentials from the Casimir operator.

At this stage we note that, in general, one can use for the construction of the symmetric UIR of $\mathfrak{so}(5)$ the carrier space with any quasi-invariant measure $\mathrm{d}\mu(x)$ on R^4. The representations with different measures are unitarily equivalent. Although the representations with different measure are mathematically equivalent, they may be related to different physical problems. For this reason, we shall consider the representation with different measures.

3. The $\mathfrak{so}(5) \supset \mathfrak{so}(4) \supset \mathfrak{so}(2) \times \mathfrak{so}(2)$ reduction

We want to diagonalize $\mathfrak{so}(4)$ and $\mathfrak{so}(2) \times \mathfrak{so}(2)$ subalgebra. Then, the reduction conditions are

$$C^{\mathfrak{so}(4)}|lmn\rangle = l(l+2)|lmn\rangle \tag{3.1}$$

$$L_{12}|lmn\rangle = m|lmn\rangle \tag{3.2}$$

$$L_{34}|lmn\rangle = n|lmn\rangle, \tag{3.3}$$

where

$$C^{\mathfrak{so}(4)} = \frac{1}{2}\sum_{i,j=1}^{4} L_{ij}^2. \tag{3.4}$$

Since $C^{\mathfrak{so}(4)}$, L_{12} and L_{34} are sought to be diagonal, we introduce in place of x_1, x_2, x_3, x_4 the variables $r, \theta, \varphi, \beta$ via

$$x_1 = r\sin\theta\sin\varphi \qquad x_2 = r\sin\theta\cos\varphi$$
$$x_3 = r\cos\theta\sin\beta \qquad x_4 = r\cos\theta\cos\beta$$

with $0 \leqslant \theta < \pi/2$, $0 \leqslant \varphi, \beta < 2\pi$ and $\mathrm{d}^4x = r^3\sin\theta\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}\beta$. Then

$$C^{\mathfrak{so}(4)} = -\left(\frac{1}{\sin\theta\cos\theta}\frac{\partial}{\partial\theta}\sin\theta\cos\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\cos^2\theta}\frac{\partial^2}{\partial\beta^2}\right) \tag{3.5}$$

$$L_{12} = \mathrm{i}\frac{\partial}{\partial\varphi}, \qquad L_{34} = \mathrm{i}\frac{\partial}{\partial\beta}. \tag{3.6}$$

If we compute the operator $\gamma^2/(C+9/4)$ for this parametrization, it becomes

$$\frac{\gamma^2}{(C+9/4)} = \frac{1}{r^3}\frac{\partial}{\partial r}r^3\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin\theta\cos\theta}\frac{\partial}{\partial\theta}\sin\theta\cos\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\cos^2\theta}\frac{\partial^2}{\partial\beta^2}\right) + \frac{2\gamma}{r}. \tag{3.7}$$

As pointed out above, the representations constructed with different measures may be related to different physical problems. Below we construct the representation in the Hilbert space $\mathfrak{H}'$ with the measure $\mathrm{d}\mu(x) = r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}\beta$. This representation, of course, is unitarily equivalent to the representation constructed in $\mathfrak{H}$. The unitary mapping W which realizes the equivalence is given by

$$W: \quad \Phi \to \Phi' = (r\cos\theta)^{1/2}\Phi. \tag{3.8}$$

That is, for the representation constructed in $\mathfrak{H}'$ the Casimir operator, call it C', is obtained by

$$C \to C' = (r\cos\theta)^{1/2} \circ C \circ (r\cos\theta)^{-1/2}, \tag{3.9}$$

where $\circ$ denotes composition of operators. Hence

$$\frac{\gamma^2}{(C'+9/4)} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) + \frac{1}{r^2\cos^2\theta}\left(\frac{1}{4} + \frac{\partial^2}{\partial\beta^2}\right) + \frac{2\gamma}{r}. \tag{3.10}$$

Let $\mathcal{H}^{(1)}$ be a subspace spanned by $|lmn\rangle$ with fixed n. Then the operator (3.10) restricted to $\mathcal{H}^{(1)}$ becomes a differential operator in r, θ, φ; it is found that

$$\left.\frac{\gamma^2}{(C'+9/4)}\right|_{\mathcal{H}^{(1)}} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) + \frac{\frac{1}{4}-n^2}{r^2\cos^2\theta} + \frac{2\gamma}{r}. \tag{3.11}$$

Hence the Hamiltonian

$$H^{(1)} = -\frac{1}{2}\nabla^2 - \frac{\gamma}{r} + \frac{n^2-\frac{1}{4}}{2r^2\cos^2\theta}, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{3.12}$$

is related to $\mathfrak{so}(5)$ in the sense that the following relation holds:

$$H^{(1)} = -\left.\frac{\gamma^2}{2(C'+9/4)}\right|_{\mathcal{H}^{(1)}}. \tag{3.13}$$

Due to the extra integral of motion

$$\widetilde{L^2} = \mathbf{L}^2 + \frac{n^2-\frac{1}{4}}{\cos^2\theta}, \tag{3.14}$$

where $\mathbf{L}^2$ is the square of angular momentum,

$$\mathbf{L}^2 = -\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right),$$

the Hamiltonian (3.12) is separable in the spherical coordinate system. Moreover, it is not difficult to see that $\widetilde{L^2}$ is related to $C'^{\mathfrak{so}(4)}$

$$\widetilde{L^2} = C'^{\mathfrak{so}(4)}|_{\mathcal{H}^{(1)}}, \tag{3.15}$$

where $C'^{\mathfrak{so}(4)} = (\cos\theta)^{1/2} \circ C^{\mathfrak{so}(4)} \circ (\cos\theta)^{-1/2}$ and $C^{\mathfrak{so}(4)}$ is given by (3.5). The second integral of motion is, of course, $L_z = \mathrm{i}(\partial/\partial\varphi)$ (due to azimuthal symmetry).

The bound-state energy spectrum can now be obtained easily if we note that the eigenvalue of C is $j(j+3)$, where j takes on integer values from zero up. Thus we find

$$E^{(1)} = -\frac{\gamma^2}{2(j+3/2)^2}, \qquad j = 0, 1, 2, \ldots. \tag{3.16}$$

Finally, we give for reference the expression for the bound-state wave functions

$$\psi(x) = \mathcal{R}_{jl}(r)\mathcal{Y}^{(1)}_{lmn}(\theta, \varphi), \tag{3.17}$$

where $\mathcal{R}_{jl}(r)$ is the radial part of the wave functions while $\mathcal{Y}^{(1)}_{lmn}(\theta,\varphi)$ is the angular part of it

$$\mathcal{R}_{jl}(r) = \kappa\, \mathrm{e}^{-\frac{u}{2}} u^{l+1/2} L^{2l+2}_{j-l}(u), \qquad u = 2\gamma r/(j+3/2) \tag{3.18}$$

$$\mathcal{Y}^{(1)}_{lmn}(\theta,\varphi) = \chi^{(1)} \sin^m\theta \cos^{n+\frac{1}{2}}\theta \mathcal{P}^{(m,n)}_k(\cos 2\theta)\exp(-\mathrm{i}m\varphi), \tag{3.19}$$

with $2k = l - m - n$. Here $L^\alpha_n(x)$ and $\mathcal{P}^{(\alpha,\beta)}_n(t)$ are Laguerre and Jacobi polynomials [12], respectively. The normalization constants κ and $\chi^{(1)}$ are given by

$$\kappa = \left[\frac{4\gamma^3\Gamma(j-l+1)}{\Gamma(j+l+3)}\right]^{\frac{1}{2}} \tag{3.20}$$

$$\chi^{(1)} = \left[\frac{(m+n+k)!k!(2k+m+n+1)}{\pi(m+k)!(n+k)!}\right]^{\frac{1}{2}}. \tag{3.21}$$

Observe that the angle function $\mathcal{Y}^{(1)}_{lmn}(\theta,\varphi)$ depend on the details of the dynamics. This is a result of very general properties, shared by all non-central Hamiltonians. It is also worth noting that the functions $\mathcal{Y}^{(1)}_{lmn}(\theta,\varphi)$ are related to matrix elements of class 1 representations of $\mathfrak{so}(4)$ in the bases corresponding to the $\mathfrak{so}(4) \supset \mathfrak{so}(2)\times\mathfrak{so}(2)$ reduction [13].

4. The $\mathfrak{so}(5) \supset \mathfrak{so}(4) \supset \mathfrak{so}(3) \supset \mathfrak{so}(2)$ reduction

Now, the reduction conditions are

$$C^{\mathfrak{so}(4))}|lsn\rangle = l(l+2)|lsn\rangle \tag{4.1}$$

$$C^{\mathfrak{so}(3)}|lsn\rangle = s(s+1)|lsn\rangle \tag{4.2}$$

$$L_{12}|lsn\rangle = n|lsn\rangle, \tag{4.3}$$

where

$$C^{\mathfrak{so}(4)} = \frac{1}{2}\sum_{i,j=1}^{4} L^2_{ij}, \qquad C^{\mathfrak{so}(3)} = \frac{1}{2}\sum_{i,j=1}^{3} L^2_{ij}. \tag{4.4}$$

The parametrization that we see for x_1, x_2, x_3, x_4 must be such as to make $C^{\mathfrak{so}(4)}$, $C^{\mathfrak{so}(3)}$ and L_{12} particularly simple,

$$x_1 = r\sin\theta\sin\varphi\sin\beta \qquad x_2 = r\sin\theta\sin\varphi\cos\beta$$
$$x_3 = r\sin\theta\cos\varphi \qquad x_4 = r\cos\theta$$

with $0 \leqslant \theta, \varphi < \pi$, $0 \leqslant \beta < 2\pi$ and $\mathrm{d}^4x = r^3\sin^2\theta\sin\varphi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}\beta$. Then

$$C^{\mathfrak{so}(4)} = -\left(\frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\sin^2\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta\sin\varphi}\frac{\partial}{\partial\varphi}\sin\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta\sin^2\varphi}\frac{\partial^2}{\partial\beta^2}\right)$$

$$C^{\mathfrak{so}(3)} = -\left(\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\sin\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\beta^2}\right)$$

$$L_{12} = \mathrm{i}\frac{\partial}{\partial\beta}$$

while

$$\frac{\gamma^2}{(C+9/4)} = \frac{1}{r^3}\frac{\partial}{\partial r}r^3\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\sin^2\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta\sin\varphi}\frac{\partial}{\partial\varphi}\sin\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta\sin^2\varphi}\frac{\partial^2}{\partial\beta^2}\right) + \frac{2\gamma}{r}. \tag{4.5}$$

The unitary mapping

$$W: \quad \Phi \to \Phi' = (r \sin\theta \sin\varphi)^{1/2}\Phi \tag{4.6}$$

brings equation (4.5) to the form

$$\frac{\gamma^2}{(C'+9/4)} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) + \frac{1}{r^2\sin^2\theta\sin^2\varphi}\left(\frac{1}{4} + \frac{\partial^2}{\partial\beta^2}\right) + \frac{2\gamma}{r}, \tag{4.7}$$

where $C' = (r\sin\theta\sin\varphi)^{1/2} \circ C \circ (r\sin\theta\sin\varphi)^{-1/2}$. Hence, the restriction of C' to a subspace $\mathcal{H}^{(2)}$ spanned by $|lsn\rangle$, for given n yields

$$\left.\frac{\gamma^2}{(C'+9/4)}\right|_{\mathcal{H}^{(2)}} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) + \frac{\frac{1}{4}-n^2}{r^2\sin^2\theta\sin^2\varphi} + \frac{2\gamma}{r}. \tag{4.8}$$

Hence the Hamiltonian

$$H^{(2)} = -\frac{1}{2}\nabla^2 - \frac{\gamma}{r} + \frac{n^2-\frac{1}{4}}{2r^2\sin^2\theta\sin^2\varphi}, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{4.9}$$

is related to $\mathfrak{so}(5)$ in the sense that the following relation holds:

$$H^{(2)} = -\left.\frac{\gamma^2}{2(C'+9/4)}\right|_{\mathcal{H}^{(2)}}. \tag{4.10}$$

In this case, the operators

$$\widetilde{L^2} = \mathbf{L}^2 + \frac{n^2-\frac{1}{4}}{\sin^2\theta\sin^2\varphi} \tag{4.11}$$

and

$$\widetilde{L_z^2} = L_z^2 + \frac{n^2-\frac{1}{4}}{\sin^2\varphi} - \frac{1}{4} \tag{4.12}$$

are responsible for separability of $H^{(2)}$ in the spherical coordinate. Moreover, it is not difficult to see that $\widetilde{L^2}$ and $\widetilde{L_z^2}$ are related to $C'^{\mathfrak{so}(4)}$ and $C'^{\mathfrak{so}(3)}$ in the sense that

$$\widetilde{L^2} = C'^{\mathfrak{so}(4)}|_{\mathcal{H}^{(2)}}, \qquad \widetilde{L_z^2} = C'^{\mathfrak{so}(3)}|_{\mathcal{H}^{(2)}}, \tag{4.13}$$

where

$$C'^{\mathfrak{so}(4)} = (\sin\theta\sin\varphi)^{1/2} \circ C^{\mathfrak{so}(4)} \circ (\sin\theta\sin\varphi)^{-1/2},$$
$$C'^{\mathfrak{so}(3)} = (\sin\varphi)^{1/2} \circ C^{\mathfrak{so}(3)} \circ (\sin\varphi)^{-1/2}.$$

The energy eigenvalues $E^{(2)}$ and their corresponding eigenfunctions $\psi^{(2)}$ are given by

$$E^{(2)} = -\frac{\gamma^2}{2(j+3/2)^2}, \qquad j = 0, 1, 2, \ldots \tag{4.14}$$

and

$$\psi^{(2)}(x) = \mathcal{R}_{jl}(r)\mathcal{Y}^{(2)}_{lsn}(\theta,\varphi), \tag{4.15}$$

where $\mathcal{R}_{jl}(r)$ is given by (3.16), while the angle function $\mathcal{Y}^{(2)}_{lsn}(\theta,\varphi)$ is given by

$$\mathcal{Y}^{(2)}_{lsn}(\theta,\varphi) = \chi^{(2)} \sin^{s+\frac{1}{2}}\theta \sin^{n+\frac{1}{2}}\varphi\, C^{1+s}_{l-s}(\cos\theta) C^{\frac{1}{2}+n}_{s-n}(\cos\varphi), \tag{4.16}$$

with the normalization constant

$$\chi^{(2)} = \left[\frac{2^{2s+2n}(l-s)!(s-n)!(s!)^2\Gamma^2\left(\frac{1}{2}+n\right)(1+l)(1+2s)}{(l+s+1)!(s+n)!}\right]^{\frac{1}{2}}. \tag{4.17}$$

Here $C_n^\lambda(t)$ is the Gegenbauer polynomials [12]. The functions $\mathcal{Y}_{lsn}^{(2)}(\theta,\varphi)$ are related to matrix elements of class 1 representations of $\mathfrak{so}(4)$ in the bases corresponding to the $\mathfrak{so}(4) \supset \mathfrak{so}(3) \supset \mathfrak{so}(2)$ reduction [13].

5. Concluding remarks

In this paper, we have investigated the nonrelativistic bound-state problem with non-central potentials using the algebraic approach. It must be noted that the potentials under consideration belong to the class of potentials admitting the separation of variables in several coordinate systems [14]. The bound-state and scattering problems for these potentials have been investigated by the path integral method in [15] and [16], respectively. Our main contribution to the solution of the problem is twofold. The first is the introduction of the potential algebra $\mathfrak{so}(5)$ which, to the best of our knowledge, has never been treated before. The second is the interrelation between extra integrals of motion responsible for separability and invariants of subalgebras of $\mathfrak{so}(5)$. Although in this paper we consider only the bound-state problem, the scattering problem can be also solved within the framework of the potential algebra, without explicit knowledge of the wave function. In this case, $\mathfrak{so}(4, 1)$ plays the role of the potential algebra. (For the approach and its implementation on some examples in one and three dimensions one may consult the papers in [17].) This will be explicitly shown in a forthcoming paper.

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